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Bipartite variation of the cheesecake factory problem: $mH(k, 2l+1)$-factorization of $K_{n,n}$

V. Chitra, A. Muthusamy *

Department of Mathematics, Periyar University, Salem - 636 011, Tamil Nadu, India

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ABSTRACT

For $k \geq 2l+1 \geq 3$, let $H(k, 2l+1)$ be a bipartite graph with bipartition $X = \{x_1, x_2, \ldots, x_k\}$, $Y = \{y_1, y_2, \ldots, y_k\}$ and edge set $\{x_i y_{i\pm j} \mid 1 \leq i \pm j \leq k, i = 1, 2, \ldots, k; j = 0, 1, 2, \ldots, l\}$. In 2009, Dalibor Froncek raised the Rectangular Table Negotiation Problem: In graph theoretical terms it is equivalent to finding an $mH(k, 2l+1)$-factorization of $K_{n,n}$, where $n = mk$. Also he answered the above problem for $H(k, 3)$ when k is odd and left open the remaining cases. In this paper, we show that the necessary conditions $n = mk$ and $m \equiv 0 (\mathrm{mod}\ \frac{\varepsilon(k,l)}{d})$, where $\varepsilon(k, l) = k(2l+1) - l(l+1) =$ the number of edges in $H(k, 2l+1)$ and $d = \gcd(k^2, \varepsilon(k, l))$ for the existence of $mH(k, 2l+1)$-factorization of $K_{n,n}$ are also sufficient when $d = 1, 2, l$, or $l+1 \equiv 0 (\mathrm{mod}\ d)$. In fact our results partially answer the Rectangular Table Negotiation Problem and also deduce the result of Froncek as a corollary.

1. Introduction

Let G, H be two simple graphs. We say that a graph G has a *H-decomposition* if there are subgraphs $H_0, H_1, H_2, \ldots, H_s$ of G, all isomorphic to H, such that each edge of G belongs to exactly one H_i for $i = 0, 1, 2, \ldots, s$; if H is a spanning subgraph of G, then we say that G has a *H-factorization*, in notation $H \parallel G$. The notation $H \nparallel G$ denotes H does not factorize G. We write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_s$, if $H_1, H_2, \ldots, H_s$ are edge disjoint subgraphs of G and $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k)$. For $k \geq 2l+1 \geq 3$, let $H(k, 2l+1)$ be a bipartite graph with bipartition $X = \{x_1, x_2, \ldots, x_k\}$, $Y = \{y_1, y_2, \ldots, y_k\}$ and edge set $\{x_i y_{i\pm j} \mid 1 \leq i \pm j \leq k,\ i = 1, 2, \ldots, k;\ j = 0, 1, 2, \ldots, l\}$. Throughout the paper, we use the parameter $\varepsilon(k, l)$ for $k(2l+1) - l(l+1)$, the number of edges in $H(k, 2l+1)$. The notation $G[U]$ denotes the subgraph of G induced by the subset $U \subseteq V(G)$. Notations and definitions not mentioned here can be seen in [1].

Let G be a bipartite graph with bipartition (V_0, V_1) and ϵ edges such that $|V_0| \leq |V_1| \leq \epsilon$. Let $\lambda : V_i \to \{0_i, 1_i, 2_i, \ldots, (\epsilon-1)_i\}$, $i = 0, 1$ be an injective map. For any pair of vertices $x \in V_0$ and $y \in V_1$ with $\lambda(x) = a_0$ and $\lambda(y) = b_1$, we define the *length* of an edge xy as $L(xy) = (b-a)(\mathrm{mod}\ \epsilon)$. We say that G has a *bigraceful labeling* if $\{L(xy) | xy \in E(G)\} = \{0, 1, 2, \ldots, \epsilon - 1\}$. It is interesting to note that, if a bipartite graph G with ϵ edges has a bigraceful labeling, then there exists a G-decomposition in $K_{\epsilon,\epsilon}$ by appropriately shifting G ϵ times.

Rectangular Table Negotiation Problem [RTNP] [2]: find a seating arrangement for two groups of $n(=mk)$ persons each around m rectangular tables of seating capacity $2k$ each, such that there are k persons from the same group along each of the long sides of the table (it is reasonable to assume that the people sitting along short sides of the table may not easily communicate with each other) and every person in one group has a conversation with every person in the other group

* Corresponding author.
E-mail addresses: chit_krish@rediffmail.com (V. Chitra), ambdu@yahoo.com (A. Muthusamy).

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exactly once over the course of t nights provided that a person can talk with

1. the person sitting directly across the table, and
2. immediate l persons (both left and right) of the person sitting directly across the table.

In terms of graph theory, the existence of such a seating arrangement is equivalent to the existence of an $mH(k, 2l+1)$-factorization of $K_{mk,mk}$, where $mH(k, 2l+1)$ is the graph consisting of m disjoint copies of $H(k, 2l+1)$ and $H(k, 2l+1)$ is the graph corresponding to the seating arrangement of one rectangular table for a night. Originally the problem was raised by Dalibor Froncek [2] and he completely solved the existence of $mH(k, 3)$-factorization of $K_{mk,mk}$ for odd k and left open the other cases.

The Rectangular Table Negotiation Problem is nothing but the bipartite variation of the Cheesecake Factory Problem [3]. In graph theoretical terms, it is equivalent to asking for a decomposition of the complete graph into the graph arising from $H(k, 3)$ by adding a path with k vertices in each partite set with the end vertices x_1, x_k and y_1, y_k respectively. The general version seems to be more difficult, and only very partial results are known.

In this paper, first we obtain necessary conditions for the existence of an $mH(k, 2l+1)$-factorization of $K_{n,n}$. Later, we show that the necessary conditions $n = mk$ and $m \equiv 0 \left(\mathrm{mod}\, \frac{\varepsilon(k,l)}{d}\right)$, $d = \gcd(k^2, \varepsilon(k, l))$ are also sufficient for the existence of $mH(k, 2l+1)$-factorization of $K_{n,n}$ when $d = 1, 2, l,$ or $l + 1 \equiv 0 \pmod d$. In fact, our results partially answer the RTNP and also deduce the earlier results of Dalibor Froncek [2] as a corollary.

2. Factorization of $K_{n,n}$ into $mH(k, 2l+1)$-factors

Lemma 2.1. *For $k \geq 2l+1 \geq 3$, let G be a graph isomorphic to $mH(k, 2l+1)$. If $K_{n,n}$ has a G-factorization, then $n = mk$ and $m \equiv 0 \left(\mathrm{mod}\, \frac{\varepsilon(k,l)}{d}\right)$, where $d = \gcd(k^2, \varepsilon(k, l))$.*

Proof. The equality $n = mk$ is obvious. To prove $m \equiv 0 \left(\mathrm{mod}\, \frac{\varepsilon(k,l)}{d}\right)$, where $d = \gcd(k^2, \varepsilon(k, l))$, we first observe that $H(k, 2l+1)$ has $\varepsilon(k, l)$ edges and therefore $m\varepsilon(k, l) | n^2$. Thus

$$n^2 = m^2k^2 = m\varepsilon(k, l)q \tag{1}$$

for some positive integer q, the number of isomorphic factors in the G-factorization. Dividing both sides of the equality by m, we get

$$mk^2 = \varepsilon(k, l)q. \tag{2}$$

Let $d = \gcd(k^2, \varepsilon(k, l))$, then $\gcd\left(\frac{k^2}{d}, \frac{\varepsilon(k,l)}{d}\right) = 1$. Therefore (2) immediately implies that

$$m \equiv 0 \left(\mathrm{mod}\, \frac{\varepsilon(k, l)}{d}\right). \quad \square$$

Observation 2.2. *If $k^2 = 2\varepsilon(k, l)$ then $H(k, 2l+1) \nparallel K_{k,k}$, since after removing one $H(k, 2l+1)$-factor from $K_{k,k}$ the resultant graph does not satisfy the required degree conditions, though it contains the right number of edges.*

Example 2.3. $H(k, 2l+1) \nparallel K_{k,k}$ when

(i) $k = 12, l = 3$;
(ii) $k = 70, l = 20$;
(iii) $k = 408, l = 119$;
(iv) $k = 2378, l = 696$.

Lemma 2.4. *For $k \geq 2l+1 \geq 3$, let $m = \varepsilon(k, l)$. Then $K_{m,m}$ has an $H(k, 2l+1)$-decomposition.*

Proof. Let the partite sets of $H = H(k, 2l+1)$ be $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$. Define map $\lambda : V(H) \to \{0, 1, 2, \ldots, \varepsilon(k, l) - 1\}$ on H as follows:

$$\lambda(x_s) = \begin{cases} 0, & s = 1; \\ (s-1)l + \binom{s}{2}, & s = 2, 3, \ldots, l; \\ \dfrac{3l(l-1)}{2} + 2l(s-l), & s = l+1, l+2, \ldots, k-l+1; \\ \dfrac{3l(l-1)}{2} + 2l(s-l) - \dbinom{s-k+l}{2}, & s = k-l+2, k-l+3, \ldots, k. \end{cases}$$

$$\lambda(y_s) = \varepsilon(k, l) - s, \quad s = 1, 2, \ldots, k.$$

Now define ϕ_j on H as $\phi_j(x_s) = (\lambda(x_s)+j)_0$, $\phi_j(y_s) = (\lambda(y_s)+j)_1$, where $0 \le j \le m-1$ and the additions are taken modulo m. Then $\{H_0, H_1, \ldots, H_{m-1}\}$, where $H_j = \phi_j(H)$, $0 \le j \le m-1$, gives an $H(k, 2l+1)$-decomposition of $K_{m,m}$. Note that for each $j = 0, 1, 2, \ldots, m-1$, ϕ_j is a bigraceful labeling of H. □

Observation 2.5. *One can observe from Lemma 2.4 that, in the $H(k, 2l+1)$-decomposition of $K_{m,m}$, where $m = \varepsilon(k, l)$, every vertex of $K_{m,m}$ appears in k copies of $H(k, 2l+1)$ and each time it appears as an image of a different vertex of $H(k, 2l+1)$ under ϕ_j.*

Theorem 2.6. *For $k \ge (2l+1) \ge 3$, let $m = \varepsilon(k, l)$, $n = km$ and $G = mH(k, 2l+1)$. Then $K_{n,n}$ has a G-factorization.*

Proof. The proof goes as follows: let $V(K_{mk,mk}) = \{(u, b)_i \mid 0 \le u \le m-1, 0 \le b \le k-1, 0 \le i \le 1\}$. By placing the subgraphs $\{H_0, H_1, \ldots, H_{m-1}\}$ obtained in the $H(k, 2l+1)$-decomposition of $K_{m,m}$ in $K_{km,km}$ appropriately, we get the base G-factor of $K_{km,km}$. By rotating the base G-factor appropriately we get a G-factorization of $K_{km,km}$.

Now we describe the G-factorization of $K_{n,n}$ as follows.

Step 1: Construction of base factor.

Let the partite sets of $H = H(k, 2l+1)$ be $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$. For each $t = 0, 1, 2, \ldots, m-1$, we define σ_t on H as

$$\sigma_t(x_s) = (\lambda(x_s) + t, s-1)_0,$$
$$\sigma_t(y_s) = (\lambda(y_s) + t, s-1)_1,$$

where λ is the map as defined in Lemma 2.4. Clearly, σ_t, $0 \le t \le m-1$ is a bigraceful labeling of H with respect to the first coordinate and $\{\sigma_0(H), \sigma_1(H), \ldots, \sigma_{m-1}(H)\}$ gives an H-decomposition of $K_{m,m}$ as the set $\{\sigma_0(H), \sigma_1(H), \ldots, \sigma_{m-1}(H)\}$ contains m edges of dimension $(u, s-1)$ for every u, $0 \le u \le m-1$ with different first coordinates. Then $G = \{\sigma_0(H), \sigma_1(H), \ldots, \sigma_{m-1}(H)\}$ is an mH-factor of $K_{n,n}$. Now we call *G the base factor* of $K_{n,n}$.

Step 2: Construction of a G-factorization from the base factor G.

We set $\mathcal{G} = \{G_{p,q} | 0 \le p,\ q \le k-1, \}$, where $G_{p,q} = \psi_{p,q}(G)$ with

$$\psi_{p,q}((u, g)_0) = (u, g+p)_0,$$
$$\psi_{p,q}((v, h)_1) = (v, h+q)_1,$$

where the additions are taken modulo k. It is easy to observe that each $G_{p,q}$ is a G-factor and contains exactly one edge $(u, g)_0(v, h)_1$ between the k-tuples $((u, 0)_0, (u, 1)_0, \ldots, (u, k-1)_0)$ and $((v, 0)_1, (v, 1)_1, \ldots, (v, k-1)_1)$, $0 \le u, v \le m-1$, $0 \le g,\ h \le k-1$. Thus $k^2 G_{p,q}$ exhaust all the k^2 edges of $K_{k,k}$ induced by the pair of k-tuples $((u, 0)_0, (u, 1)_0, \ldots, (u, k-1)_0)$ and $((v, 0)_1, (v, 1)_1, \ldots, (v, k-1)_1)$, $0 \le u,\ v \le m-1$. Therefore, every edge of $K_{n,n}$, where $n = km$ appears exactly in one of the $G_{p,q}$. Thus $\mathcal{G}$ gives our required factorization. □

Remark 2.7. If $K_{n,n}$ has a G-factorization then $K_{rn,rn}$ has a G-factorization for all $r \ge 1$.

Combining Theorem 2.6 and Remark 2.7, we have

Corollary 2.8. *For $k \ge (2l+1) \ge 3$, let $m \equiv 0 \pmod{\varepsilon(k, l)}$ and $n = mk$. Then $K_{n,n}$ has a G-factorization, where $G = mH(k, 2l+1)$.*

Note 2.9. Corollary 2.8 shows that the necessary conditions stated in Lemma 2.1 are also sufficient when $d = (k^2, \varepsilon(k, l)) = 1$. In fact, Corollary 2.8 deduce the result of Dalibor Froncek [2] when $l = 1$.

Lemma 2.10. *For a given $d|k$, where $d = \gcd(k^2, \varepsilon(k, l))$, $k \ge 2l+1 \ge 3$, if the bipartite graph $H(k, 2l+1)$ with bipartition $(X = \{x_1, x_2, x_3, \ldots, x_k\},\ Y = \{y_1, y_2, y_3, \ldots, y_k\})$ has a finite collection $\{X_i\}_{i=1}^{d}$, where $X_i = \left\{x_{i_1}, x_{i_2}, \ldots, x_{i_{\frac{k}{d}}}\right\} \subset X$, $i_1 < i_2 < \cdots < i_{\frac{k}{d}}$ such that*

(i) *$X_i \cap X_j = \emptyset$ for all $i \ne j$,*
(ii) *$\bigcup_{i=1}^{d} X_i = X$,*
(iii) *$\sum_{x_{i_j} \in X_i} \deg(x_{i_j}) = \frac{\varepsilon(k,l)}{d} = p(say)$ and $N(x_{i_j}) \cap N(x_{i_{j+1}}) \ne \emptyset$ for $i = 1, 2, \ldots, d$*

then $H(k, 2l+1)$ admits a bigraceful labeling λ such that the edges of the subgraph induced by $[X_i, Y]$, $i = 1, 2, \ldots, d$ have lengths from $(i-1)p$ to $ip-1$.

Proof. Without loss of generality, assume that $x_{1_1} = x_1 \in X_1$. By the hypothesis, we have $H(k, 2l+1) = H[X_1, Y] \oplus H[X_2, Y] \oplus \cdots \oplus H[X_d, Y]$ and $|E(H[X_i, Y])| = p$ for all $i = 1, 2, \ldots, d$. In $H(k, 2l+1)$, let $y_{2(1)}, y_{3(1)}, \ldots, y_{d(1)}$ be the leftmost neighbors of $x_{2_1}, x_{3_1}, \ldots, x_{d_1}$ respectively.

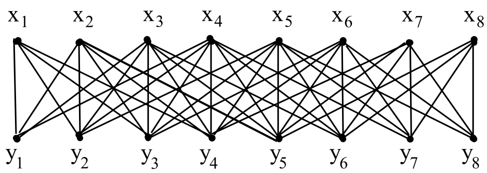

Fig. 1. $H(8, 7)$.

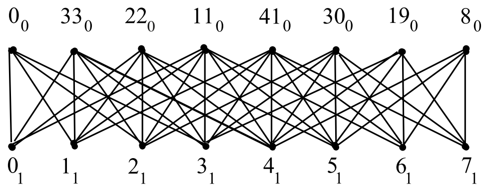

Fig. 2. Bigraceful labeling of $H(8, 7)$ when $X_1 = \{x_1, x_5\}, X_2 = \{x_2, x_6\}, X_3 = \{x_3, x_7\}, X_4 = \{x_4, x_8\}$.

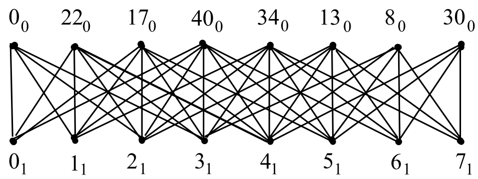

Fig. 3. Bigraceful labeling of $H(8, 7)$ when $X_1 = \{x_1, x_4\}, X_2 = \{x_5, x_8\}, X_3 = \{x_2, x_3\}, X_4 = \{x_6, x_7\}$.

Construct a bigraceful labeling $\lambda : X \cup Y \to \{0_0, 1_0, 2_0, \dots, \varepsilon(k, l)_0\} \cup \{0_1, 1_1, 2_1, \dots, \varepsilon(k, l)_1\}$ such that the edges of the subgraph induced by $[X_i, Y]$, $i = 1, 2, \dots, d$ have lengths from $(i - 1)p$ to $ip - 1$ as follows:

$$\lambda(y_a) = (a - 1)_1, \quad 1 \le a \le k.$$

For $i = 1,\ j = 1, 2, \dots, \frac{k}{d}$

$$\lambda(x_{1_j}) = \begin{cases} 0_0, & j = 1; \\ \left(\varepsilon(k, l) - \sum_{s=1}^{j-1} (|N(x_{i_s}) \cap N(x_{i_{s+1}})|)\right)_0, & j = 2, 3, \dots, \dfrac{k}{d}. \end{cases}$$

For $i = 2, 3, \dots, d,\ j = 1, 2, \dots, \frac{k}{d}$

$$\lambda(x_{i_j}) = \begin{cases} (\xi_{i(1)})_0, & j = 1; \\ \left(\lambda(x_{i_1}) - \sum_{s=1}^{j-1} (|N(x_{i_s}) \cap N(x_{i_{s+1}})|)\right)_0, & j = 2, 3, \dots, \dfrac{k}{d} \end{cases}$$

where $\xi_{i(1)}$ is a unique integer from the set $\{0, 1, 2, \dots, \varepsilon(k, l)\}$ such that

(i) $\lambda(y_{i(1)}) - \xi_{i(1)} \equiv (i - 1)p (\mathrm{mod}\ \varepsilon(k, l))$ and
(ii) $\xi_{i(1)} \ne \lambda(x_{\alpha_j})$, $\alpha = 1, 2, 3, \dots, i - 1$, $j = 1, 2, \dots, \frac{k}{d}$. □

Example 2.11. Bigraceful labeling of $H(8, 7)$ (see Fig. 1 for $H(8, 7)$):
Here $y_1, y_1, y_1, y_1, y_2, y_3, y_4, y_5$ are the leftmost neighbors of $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ respectively. Further, it is easy to see that the following sets satisfy the hypothesis of Lemma 2.10.

1. $X_1 = \{x_1, x_5\}, X_2 = \{x_2, x_6\}, X_3 = \{x_3, x_7\}, X_4 = \{x_4, x_8\}$, or
2. $X_1 = \{x_1, x_4\}, X_2 = \{x_5, x_8\}, X_3 = \{x_2, x_3\}, X_4 = \{x_6, x_7\}$, or
3. $X_1 = \{x_1, x_5\}, X_2 = \{x_2, x_3\}, X_3 = \{x_4, x_8\}, X_4 = \{x_6, x_7\}$.

The bigraceful labeling of $H(8, 7)$ as in Lemma 2.10 for the above three possibilities of sets are shown in Figs. 2–4.

Theorem 2.12. *For a given $d|k$, where $d = \gcd(k^2, \varepsilon(k, l))$, $k \ge 2l + 1 \ge 3$, if $H(k, 2l + 1)$ satisfies the hypothesis of Lemma 2.10, then $K_{pk,pk}$ has an $pH(k, 2l + 1)$-factorization, where $p = \frac{\varepsilon(k,l)}{d}$.*

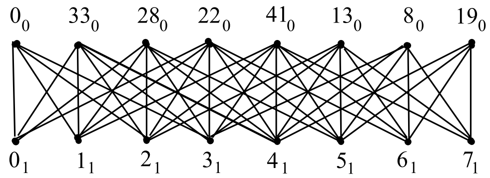

Fig. 4. Bigraceful labeling of $H(8, 7)$ when $X_1 = \{x_1, x_5\}, X_2 = \{x_2, x_3\}, X_3 = \{x_4, x_8\}, X_4 = \{x_6, x_7\}$.

Proof. Let $V(K_{pk,pk}) = \left\{(u, i_j)_0 : 0 \le u \le p-1, 1 \le i \le d, 1 \le j \le \frac{k}{d}\right\} \cup \{(u, t)_1 : 0 \le u \le p-1, 1 \le t \le k\}$.

By the hypothesis, we have $H(k, 2l+1) = H[X_1, Y] \oplus H[X_2, Y] \oplus \cdots \oplus H[X_d, Y]$ and $|E(H[X_i, Y])| = p$ for all $i = 1, 2, \ldots, d$. By Lemma 2.10 there exists a bigraceful labeling $\lambda : X \cup Y \to \{0_0, 1_0, 2_0, \ldots, \varepsilon(k, l)_0\} \cup \{0_1, 1_1, 2_1, \ldots, \varepsilon(k, l)_1\}$ for $H(k, 2l+1)$ such that the subgraphs induced by $[X_i, Y]$, $i = 1, 2, \ldots, d$ have lengths from $(i-1)p$ to $ip - 1$. We denote $H(k, 2l+1)$ with its bigraceful labeling λ as H.

Step 1: Construction of base factor:

For $t = 0, 1, \ldots, p-1$, construct a new graph $H_t = \sigma_t(H)$ with

$$\sigma_t(\lambda(x_{i_j})) = \left(c_{i_j} + t, i_j\right)_0,$$
$$\sigma_t(\lambda(y_a)) = \left(\lambda^*(y_a) + t, a\right)_1,$$

$c_{i_j} \equiv \lambda^*(x_{i_j}) (\mathrm{mod}\ p)$, where $\lambda^*(x_{i_j}), \lambda^*(y_a)$ are nothing but $\lambda(x_{i_j}), \lambda(y_a)$ without suffixes 0, 1 respectively and the additions $c_{i_j} + t, \lambda^*(y_a) + t$ are taken modulo p. Then $G = \{H_0, H_1, H_2, \ldots, H_{p-1}\}$ is a $pH(k, 2l+1)$-factor of $K_{n,n}$, where $n = kp$. Now we call G *the base factor* of $K_{n,n}$.

From the above construction, we observe that for each t the edges of H_t have lengths from 0 to $p-1$ with respect to the first coordinate, and exactly d edges have lengths a, $0 \le a \le p-1$, which is guaranteed by Lemma 2.10. Since G has $p^2 d$ edges, and there are $p^2 d$ such pairs. Therefore in the base factor G, there are exactly d edges between every pair of $\frac{k}{d}$-tuples

$$\left(\underbrace{((u, 1_1)_0, (u, 2_1)_0, \ldots, (u, d_1)_0)}_{\text{1st coordinate of } \frac{k}{d}\text{-tuple}}, \underbrace{((u, 1_2)_0, (u, 2_2)_0, \ldots, (u, d_2)_0)}_{\text{2nd coordinate of } \frac{k}{d}\text{-tuple}}, \ldots, \underbrace{\left(\left(u, 1_{\frac{k}{d}}\right)_0, \left(u, 2_{\frac{k}{d}}\right)_0, \ldots, \left(u, d_{\frac{k}{d}}\right)_0\right)}_{\frac{k}{d}\text{th coordinate of } \frac{k}{d}\text{-tuple}}\right)$$

and k-tuples $((v, 1)_1, (v, 2)_1, \ldots, (v, k)_1)$.

Step 2: Construction of a G-factorization from the base factor G:

We set $\mathcal{G} = \left\{G_{q,q'} | 0 \le q \le \frac{k}{d} - 1, 0 \le q' \le k-1\right\}$, where $G_{q,q'} = \psi_{q,q'}(G)$ with

$$\psi_{q,q'}(\sigma_t(\lambda(x_{i_j}))) = (c_{i_j} + t, i_{j+q})_0, \quad 1 \le i \le d;\ 1 \le j \le \frac{k}{d},$$
$$\psi_{q,q'}(\sigma_t(\lambda(y_a))) = (\lambda^*(y_a) + t, a + q')_1, \quad 1 \le a \le k.$$

$c_{i_j} \equiv \lambda^*(x_{i_j}) (\mathrm{mod}\ p)$, where $\lambda^*(x_{i_j}), \lambda^*(y_a)$ are nothing but $\lambda(x_{i_j}), \lambda(y_a)$ without suffixes 0, 1 respectively and the additions $c_{i_j} + t, \lambda^*(y_a) + t, j + q, a + q'$ are taken modulo $p, p, \frac{k}{d}, k$ respectively.

Next we show that the images of the edges are all distinct. For a fixed q, q', if $\psi_{q,q'}((\sigma_t(\lambda(x_{i_j})))(\sigma_t(\lambda(y_a)))) = \psi_{q,q'}(\sigma_t(\lambda(x_{i'_{j'}}))(\sigma_t(\lambda(y_b))))$ then $i = i', j = j'$ and $a = b$ since $i_1 < i_2 < \cdots < i_{\frac{k}{d}}$ for all $1 \le i \le d$ and $\left\{i_j : 1 \le i \le d, 1 \le j \le \frac{k}{d}\right\} = \{1, 2, \ldots, k\}$. Hence the images are all distinct.

Note that $G_{0,0}$ is nothing but the base $mH(k, 2l+1)$-factor. Clearly, for fixed q, $0 \le q \le \frac{k}{d} - 1$, we have $E(G_{q,r'}) \cap E(G_{q,s'}) = \emptyset$ for all $r' \ne s'$, $0 \le r', s' \le k-1$. Similarly for fixed q', $0 \le q' \le k-1$, we have $E(G_{r,q'}) \cap E(G_{s,q'}) = \emptyset$ for all $r \ne s$, $0 \le r, s \le \frac{k}{d} - 1$. Further for $q \ne r$, $q' \ne s'$, $E(G_{q,q'}) \cap E(G_{r,s'}) = \emptyset$ for all $0 \le q, r \le \frac{k}{d} - 1$, $0 \le q', s' \le k-1$.

It is easy to observe that each $G_{q,q'}$ contains exactly d edges between every pair of $\frac{k}{d}$-tuples

$$\left(\underbrace{((u, 1_1)_0, (u, 2_1)_0, \ldots, (u, d_1)_0)}_{\text{1st coordinate of } \frac{k}{d}\text{-tuple}}, \underbrace{((u, 1_2)_0, (u, 2_2)_0, \ldots, (u, d_2)_0)}_{\text{2nd coordinate of } \frac{k}{d}\text{-tuple}}, \ldots, \underbrace{\left(\left(u, 1_{\frac{k}{d}}\right)_0, \left(u, 2_{\frac{k}{d}}\right)_0, \ldots, \left(u, d_{\frac{k}{d}}\right)_0\right)}_{\frac{k}{d}\text{th coordinate of } \frac{k}{d}\text{-tuple}}\right)$$

and k-tuples $((v,1)_1,(v,2)_1,\ldots,(v,k)_1)$, and hence the $\frac{k^2}{d}$ images $\psi_{q,q'}((u,c)_0(b,d)_1)$ for $q=0,1,\ldots,\frac{k}{d}-1, q'=0,1,\ldots,k-1$ induce the complete bipartite graph $K_{\frac{k}{d},k}$ with the bipartition

$$X_u=\left\{\left(((u,1_1)_0,(u,2_1)_0,\ldots,(u,d_1)_0),((u,1_2)_0,(u,2_2)_0,\ldots,(u,d_2)_0),\ldots,\left(\left(u,1_{\frac{k}{d}}\right)_0,\left(u,2_{\frac{k}{d}}\right)_0,\ldots,\left(u,d_{\frac{k}{d}}\right)_0\right)\right)\right\},$$
$$Y_v=\{((v,0)_1,(v,1)_1,\ldots,(v,k-1)_1)\}.$$

Therefore, every edge of $K_{pk,pk}$, appears in exactly one $G_{q,q'}$. Thus $\mathcal{G}$ gives our required factorization. $\square$

Combining Theorem 2.12 and Remark 2.7, we have

Corollary 2.13. *If $H(k,2l+1)$ satisfy the hypothesis of Lemma 2.10, then $pH(k,2l+1)\parallel K_{rpk,rpk}$, $r\ge 1$.*

Theorem 2.14. *The complete bipartite graph $K_{n,n}$ has a G-factorization, where $G=mH(k,2l+1)$ and $n=km$ if*

1. $\gcd(k^2,\varepsilon(k,l))=2$ *and*
2. $m\equiv 0\left(mod\ \frac{\varepsilon(k,l)}{2}\right)$.

Proof. Let $(X=\{x_1,x_2,\ldots,x_k\}, Y=\{y_1,y_2,\ldots,y_k\})$ be the bipartition of $H(k,2l+1)$ and $X_1=\left\{x_1,x_2,\ldots,x_{\frac{k}{2}}\right\}, X_2=\left\{x_{\frac{k}{2}+1},x_{\frac{k}{2}+2},\ldots,x_k\right\}$ with $X_1,X_2\subset X$ as in Lemma 2.10. Then $\sum_{x_s\in X_1}\deg(x_s)=\sum_{x_s\in X_2}\deg(x_s)=\frac{\varepsilon(k,l)}{2}$, $N(x_i)\cap N(x_{i+1})\neq\emptyset$ and

$$\deg(x_i)=\begin{cases}l+i, & i=1,2,\ldots,l;\\ 2l+1, & i=l+1,l+2,\ldots,k-l;\\ 2l+1-(i-(k-l)), & i=k-l+1,k-l+2,\ldots,k.\end{cases}$$

Now by Lemma 2.10 there exists a bigraceful labeling $\lambda: X\cup Y\to\{0_0,1_0,2_0,\ldots,\varepsilon(k,l)_0\}\cup\{0_1,1_1,2_1,\ldots,\varepsilon(k,l)_1\}$as

$$\lambda(x_s)=\begin{cases}(0)_0, & s=1;\\ \left(\varepsilon(k,l)-(s-1)l-\binom{s}{2}\right)_0, & s=2,3,\ldots,l;\\ \left(\varepsilon(k,l)-3\binom{l}{2}-2l(s-l)\right)_0, & s=l+1,l+2,\ldots,k-l+1;\\ \left(\varepsilon(k,l)-3\binom{l}{2}-2l(s-l)+\binom{s-k+l}{2}\right)_0, & s=k-l+2,k-l+3,\ldots,k.\end{cases}$$
$$\lambda(y_s)=(s-1)_1,\quad s=1,2,\ldots,k$$

such that the edges of the subgraph induced by $[X_i,Y]$, $i=1,2$ have lengths from $(i-1)\left(\frac{\varepsilon(k,l)}{2}\right)$ to $i\left(\frac{\varepsilon(k,l)}{2}\right)-1$. As $H(k,2l+1)$ satisfies the hypothesis of Theorem 2.12, we have a G-factorization of $K_{n,n}$. $\square$

Note 2.15. Theorem 2.14 shows that the necessary conditions stated in Lemma 2.1 are also sufficient when $d=\gcd(k^2,\varepsilon(k,l))=2$ which implies that $K_{mk,mk}$ has an $mH(k,3)$-factorization, when $k\equiv 0 (\mathrm{mod}\ 4)$. In fact, Theorem 2.14 partially answers the RTNP raised by Dalibor Froncek [2].

Theorem 2.16. *The complete bipartite graph $K_{n,n}$ has a G-factorization, where $G=mH(k,2l+1)$ and $n=km$ if*

1. $\gcd(k^2,\varepsilon(k,l))=d$, $d|l$ *and* $d|k$,
2. $m\equiv 0\left(mod\ \frac{\varepsilon(k,l)}{d}\right)$.

Proof. Let $(X=\{x_1,x_2,\ldots,x_k\}, Y=\{y_1,y_2,\ldots,y_k\})$ be the bipartition of $H(k,2l+1)$ and $X_i=\{x_{i+dj};\ 0\le j\le\frac{k}{d}-1\}$, $i=1,2,\ldots,d$ with $X_i\subset X$. Then $\sum_{x_s\in X_i}\deg(x_s)=\frac{\varepsilon(k,l)}{d}$, $N(x_{i+dj})\cap N(x_{i+d(j+1)})\neq\emptyset$ and for each $i=1,2,\ldots,d$

$$\deg(x_{i+dj})=\begin{cases}l+dj+i, & j=0,1,\ldots,\frac{l}{d}-1;\\ 2l+1, & j=\frac{l}{d},\frac{l}{d}+1,\ldots,\frac{k-l}{d}-1;\\ 2l-(i-1)-\left(j-\left(\frac{k-l}{d}\right)\right)d, & j=\frac{k-l}{d},\frac{k-l}{d}+1,\ldots,\frac{k}{d}-1.\end{cases}$$

Now by Lemma 2.10 there exists a bigraceful labeling $\lambda : X \cup Y \to \{0_0, 1_0, 2_0, \ldots, \varepsilon(k,l)_0\} \cup \{0_1, 1_1, 2_1, \ldots, \varepsilon(k,l)_1\}$ as follows:

For $i = 1;\ j = 0, 1, 2, \ldots, \left(\frac{k}{d} - 1\right)$

$$\lambda(x_{1+dj}) = \begin{cases} (0)_0, \quad j = 0; \\ \left(\varepsilon(k,l) - \sum_{s=1}^{j}(l + (s-1)d + 1)\right)_0, \quad j = 1, 2, \ldots, \frac{l}{d} - 1; \\ \left(\varepsilon(k,l) - \sum_{s=1}^{\frac{l}{d}-1}(l + (s-1)d + 1) - (2l + 1 - d)\left(j + 1 - \frac{l}{d}\right)\right)_0, \quad j = \frac{l}{d}, \frac{l}{d} + 1, \frac{l}{d} + 2, \ldots, \frac{k-l}{d}; \\ \left(\varepsilon(k,l) - \sum_{s=1}^{\frac{l}{d}-1}(l + (s-1)d + 1) - (2l + 1 - d)\left(\frac{k-2l}{d} + 1\right) - \sum_{s=1}^{j-\left(\frac{k-l}{d}\right)}(2l - sd)\right)_0, \\ \quad j = \frac{k-l}{d} + 1, \frac{k-l}{d} + 2, \ldots, \frac{k}{d} - 1. \end{cases}$$

For $i = 2, 3, \ldots, d;\ j = 0, 1, 2, \ldots, \left(\frac{k}{d} - 1\right)$

$$\lambda(x_{i+dj}) = \begin{cases} (\lambda(x_{i-1+k-d}) - k)_0, & j = 0; \\ \left(\lambda(x_i) - \left(\sum_{s=1}^{j}(l + (s-1)d + i)\right)\right)_0, & j = 1, 2, \ldots, \frac{l}{d} - 1; \\ \left(\lambda(x_{i+l-d}) - (2l + 1 - d)\left(j + 1 - \frac{l}{d}\right)\right)_0, & j = \frac{l}{d}, \frac{l}{d} + 1, \frac{l}{d} + 2, \ldots, \frac{k-l}{d}; \\ \left(\lambda(x_{i+k-l}) - \left(\sum_{s=1}^{j-\left(\frac{k-l}{d}\right)}(2l - sd - i + 1)\right)\right)_0, & j = \frac{k-l}{d} + 1, \frac{k-l}{d} + 2, \ldots, \frac{k}{d} - 1. \end{cases}$$

$$\lambda(y_{i+dj}) = \left(i + dj - 1\right)_1, \quad i = 1, 2, \ldots, d; j = 0, 1, 2, \ldots, \left(\frac{k}{d} - 1\right).$$

such that the edges of the subgraph induced by $[X_i, Y],\ i = 1, 2, \ldots, d$ have lengths from $(i-1)\left(\frac{\varepsilon(k,l)}{d}\right)$ to $i\left(\frac{\varepsilon(k,l)}{d}\right) - 1$. As $H(k, 2l+1)$ satisfies the hypothesis of Theorem 2.12, we have a G-factorization of $K_{n,n}$. $\square$

Theorem 2.17. *The complete bipartite graph $K_{n,n}$ has a G-factorization, where $G = mH(k, 2l+1)$ and $n = km$ if*

1. $\gcd(k^2, \varepsilon(k,l)) = d,\ d|l+1$ *and* $d|k$,
2. $m \equiv 0 \left(mod\ \frac{\varepsilon(k,l)}{d}\right)$.

Proof. Let $(X = \{x_1, x_2, \ldots, x_k\}, Y = \{y_1, y_2, \ldots, y_k\})$ be the bipartition of $H(k, 2l+1)$ and $X_i = \{x_{i+dj};\ 0 \le j \le \frac{k}{d} - 1\},\ i = 1, 2, \ldots, d$ with $X_i \subset X$. Then $\sum_{x_s \in S_i} \deg(x_s) = \frac{\varepsilon(k,l)}{d}$, $N(x_{i+dj}) \cap N(x_{i+d(j+1)}) \neq \emptyset$ and for each $i = 1, 2, \ldots, d$

$$\deg(x_{i+dj}) = \begin{cases} l + dj + i, & j = 0, 1, \ldots, \frac{l+1}{d} - 1; \\ 2l + 1, & j = \frac{l+1}{d}, \frac{l+1}{d} + 1, \ldots, \frac{k-l-1}{d} - 1; \\ 2l + 1 - (i-1) - \left(j - \frac{k-l-1}{d}\right)d, & j = \frac{k-l-1}{d}, \frac{k-l-1}{d} + 1, \ldots, \frac{k}{d} - 1. \end{cases}$$

Now by Lemma 2.10, there exists a bigraceful labeling $\lambda : V_0 \cup V_1 \to \{0_0, 1_0, 2_0, \ldots, \varepsilon(k, l)_0\} \cup \{0_1, 1_1, 2_1, \ldots, \varepsilon(k, l)_1\}$ as follows:
For $i = 1;\ j = 0, 1, 2, \ldots, \left(\frac{k}{d} - 1\right)$

$$\lambda(x_{1+dj}) = \begin{cases} (0)_0, \quad j = 0; \\ \left(\varepsilon(k,l) - \sum_{s=1}^{j}(l + (s-1)d + 1)\right)_0, \quad j = 1, 2, \ldots \dfrac{l+1}{d} - 1; \\ \left(\varepsilon(k,l) - \sum_{s=1}^{\frac{l+1}{d}-1}(l + (s-1)d + 1) - (2l+1-d)\left(j + 1 - \dfrac{l+1}{d}\right)\right)_0, \\ \quad j = \dfrac{l+1}{d}, \dfrac{l+1}{d} + 1, \dfrac{l+1}{d} + 2, \ldots, \dfrac{k-l-1}{d}; \\ \left(\varepsilon(k,l) - \sum_{s=1}^{\frac{l+1}{d}-1}(l + (s-1)d + 1) - (2l+1-d)\left(\dfrac{k-2l-2}{d} + 1\right) - \sum_{s=1}^{j-\left(\frac{k-l-1}{d}\right)}(2l+1-sd)\right)_0, \\ \quad j = \dfrac{k-l-1}{d} + 1, \dfrac{k-l-1}{d} + 2, \ldots, \dfrac{k}{d} - 1. \end{cases}$$

For $i = 2, 3, \ldots, d;\ j = 0, 1, 2, \ldots, \left(\frac{k}{d} - 1\right)$

$$\lambda(x_{i+dj}) = \begin{cases} (\lambda(x_{i-1+k-d}) - k)_0, & j = 0; \\ \left(\lambda(x_i) - \left(\sum_{s=1}^{j}(l + (s-1)d + i)\right)\right)_0, & j = 1, 2, \ldots, \dfrac{l+1}{d} - 1; \\ \left(\lambda(x_{i+l+1-d}) - (2l+1-d)\left(j + 1 - \dfrac{l+1}{d}\right)\right)_0, & j = \dfrac{l+1}{d}, \dfrac{l+1}{d} + 1, \dfrac{l+1}{d} + 2, \ldots, \dfrac{k-l-1}{d}; \\ \left(\lambda(x_{i+k-l-1}) - \left(\sum_{s=1}^{j-\left(\frac{k-l-1}{d}\right)}(2l+2-sd-i)\right)\right)_0, & j = \dfrac{k-l-1}{d} + 1, \dfrac{k-l-1}{d} + 2, \ldots, \dfrac{k}{d} - 1. \end{cases}$$

$$\lambda(y_{i+dj}) = \left(i + dj - 1\right)_1, \quad i = 1, 2, \ldots, d;\ j = 0, 1, 2, \ldots, \left(\frac{k}{d} - 1\right).$$

such that the edges of the subgraph induced by $[X_i, Y],\ i = 1, 2, \ldots, d$ have lengths from $(i-1)\left(\frac{\varepsilon(k,l)}{d}\right)$ to $i\left(\frac{\varepsilon(k,l)}{d}\right) - 1$. As $H(k, 2l+1)$ satisfies the hypothesis of Theorem 2.12, we have a G-factorization of $K_{n,n}$. □

Note 2.18. Theorems 2.16 and 2.17 shows that the necessary conditions stated in Lemma 2.1 for the existence of $mH(k, 2l+1)$-factorization of $K_{n,n}$, $n = mk$ are also sufficient when $d \mid l$ or $d \mid (l+1)$.

3. Conclusion

In [2], Dalibor Froncek completely settled the existence of an $mH(k, 3)$-factorization of $K_{n,n}$ when k is odd. In this paper, we have proved that there exists an $mH(k, 2l+1)$-factorization of $K_{mk,mk}$ when $m \equiv 0 (\text{mod } \varepsilon(k, l))$. Further, we proved that the necessary conditions $n = mk$ and $m \equiv 0 \left(\text{mod} \frac{\varepsilon(k,l)}{d}\right),\ d = \gcd(k^2, \varepsilon(k, l))$ are also sufficient for the existence of $mH(k, 2l+1)$-factorization of $K_{n,n}$ when $d = 1, 2, l,$ or $l + 1 \equiv 0 (\text{mod } d)$. In fact, our results deduce the result of Dalibor Froncek [2] when $l = 1$ and also partially answer the RTNP raised by Dalibor Froncek [2]. The RTNP remains open for other values of d.

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